$\boxed{\text{Chapter 10 – Day 4}}$

Ex: Suppose $f(x) = \begin{cases} 2x & x \leq 2 \\ 8/x & x > 2 \end{cases}$

Evaluate $\int_0^5 f(x)\,dx$

$$\int_0^5 f(x)\,dx = \int_0^2 2x\,dx + \int_2^5 \frac{8}{x}\,dx$$

$$= (1x^2)\Big|_0^2 + (8\ln|x|)\Big|_2^5$$

$$= 2^2 - 0^2 + 8\ln 5 - 8\ln 2$$

$$= \boxed{4 + 8\ln 5 - 8\ln 2}$$

Substitution Rule for Integrals

if $u = g(t)$ is a differentiable function whose range is a subinterval I and f is continuous on I, then

$$\int f(g(t))\, g'(t)\, dt = \int f(u)\, du$$

for definite integrals

$$\int_a^b f(g(t))\, g'(t)\, dt = \int_{g(a)}^{g(b)} f(u)\, du$$

Ex: Evaluate $\int_0^x (t+9)^2\,dt$

let $u = t+9$

$\frac{du}{dt} = 1 \Rightarrow du = 1\,dt$

if $t=0$, then $u = 0+9 = 9$

if $t=x$, then $u = x+9$

then $\int_0^x (t+9)^2\,dt = \int_9^{x+9} u^2\,du$

$$= \left(\frac{1}{3}u^3\right)\Big|_9^{x+9}$$

$$= \boxed{\frac{1}{3}(x+9)^3 - \frac{1}{3}(9)^3}$$

Ex: Evaluate $\int_0^x \sqrt{3t+7}\,dt$

let $u = 3t+7$

$\frac{du}{dt} = 3 \Rightarrow du = 3\,dt$

$\frac{1}{3}du = dt$

if $t=0$ then $u = 3(0)+7 = 7$

if $t=x$ then $u = 3x+7$

$$\int_0^x \sqrt{3t+7}\,dt = \int_7^{3x+7} u^{1/2} \cdot \frac{1}{3}du$$

$$= \left(\frac{1/3}{3/2} u^{3/2}\right)\Big|_7^{3x+7}$$

$$= \boxed{\frac{2}{9}(3x+7)^{3/2} - \frac{2}{9}(7)^{3/2}}$$

Ex: Evaluate $\int_0^1 5e^{5x+1}\,dx$

let $u = 5x+1$

$$\frac{du}{dx} = 5 \Rightarrow du = 5\,dx$$

if $x=0$ then $u=1$

if $x=1$ then $u=6$

$$\int_0^1 \underbrace{5e^{5x+1}\,dx}_{du} = \int_1^6 e^u\,du$$

$$= \left(e^u\right)\Big|_1^6$$

$$= \boxed{e^6 - e^1}$$

Ex: Evaluate $\int_0^3 \frac{2x}{x^2+1}\,dx$

let $u = x^2+1$

$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

if $x=0$ then $u=1$

if $x=3$ then $u=10$

$$\int_0^3 \frac{2x}{x^2+1}\,\underbrace{dx}_{du} = \int_1^{10} \frac{1}{u}\,du$$

$$= \left(\ln|u|\right)\Big|_1^{10}$$

$$= \ln 10 - \ln 1$$

$$= \ln 10 - 0$$

$$= \boxed{\ln 10}$$

Ex: Compute the derivative of

$$F(x) = \int_0^{x^2} 2t\,dt$$

Hint: let $f(x) = x^2$ and $g(x) = \int_0^{x} 2t\,dt$

Then

$$g(f(x)) = g(x^2) = \int_0^{x^2} 2t\,dt = F(x)$$

Then by chain rule,

$$F'(x) = g'(f(x)) \cdot f'(x)$$
$$= g'(x^2) \cdot 2x$$

by FTC, $g'(x) = 2x$ so $g'(x^2) = 2x^2$

then $F'(x) = g'(x^2) \cdot 2x$

$$= 2x^2 \cdot 2x$$
$$= \boxed{4x^3}$$

In general, if $F(x) = \int_a^{f(x)} H(t)\,dt$

then $F'(x) = \frac{d}{dx} \int_a^{f(x)} H(t)\,dt$

$$= H(f(x)) \cdot f'(x)$$